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On the Construction of Wold Decomposition for Multivariate Stationary Processes

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A method to construct the Wold decomposition for multivariate stationary stochastic processes $\mathbf{x}_k$, $k \in Z$, is presented. The method is based on orthogonal decompositions for $\mathbf{x}_k$, $k \in Z$, obtained by forming orthogonal projections of $\mathbf{x}_k$, $k \in Z$, onto its component processes x_k^j, $k \in Z$, $j = 1,..., q$. The method does not give a complete solution to the Wold decomposition problem.

INTRODUCTION

In this paper we consider the construction of the Wold decomposition for a q-variate stationary stochastic process $\mathbf{x}_k$, $k \in Z$, provided that the spectral measure and, in general, even the rank of $\mathbf{x}_k$, $k \in Z$, are known. Our method is based on the forming of orthogonal decompositions for $\mathbf{x}_k$, $k \in Z$, by applying orthogonal projections of $\mathbf{x}_k$, $k \in Z$, onto its component processes x_k^j, $k \in Z$; $j = 1,..., q$, and then use criteria given by Robertson [13] and Jang Ze-Pei [2] to decide whether the result is the desired Wold decomposition. Unfortunately our method does not give a complete solution to the problem how to construct the Wold decomposition (cf. Example 9(b)).

For the sake of completeness, we present in Section 1 the needed results concerning bounded orthogonally scattered pairwise biorthogonal vector measures with values in a Hilbert space. The main results in Section 1 seem to be obtained already by Matveev [7].

1. ON BIORTHOGONAL ORTHOGONALLY SCATTERED VECTOR MEASURES

In this section we present some preliminary results concerning a finite collection of pairwise biorthogonal orthogonally scattered vector measures with

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values in a Hilbert space. The main results (Theorem 2) seem to be obtained already in a paper by Metveev [7]. The basic method to prove our results is to apply results obtained by Masani [5; Theorem 5.10] and Ressel [12; Theorem 1]. For convenience we consider here only the case of bounded orthogonally scattered vector measures.

Let S be a locally compact Hausdorff space. By $C_0(S)$ we denote the linear space of all continuous functions $f: S \to C$ vanishing at infinity carrying the sup-norm topology. Let H be a (fixed) complex Hilbert space. Recall that a bounded *orthogonally scattered* vector measure on S with values in H is a bounded linear mapping μ: $C_0(S) \to H$, or equivalently a countably additive regular set function defined on the Borel sets $\mathscr{S}$ of S, such that

$$(\mu(A) \mid \mu(B)) = 0 \qquad \text{for all} \quad A, B \in \mathscr{S};\ A \cap B = \varnothing.$$

For a bounded orthogonally scattered vector measure μ, there exists a uniquely determined bounded positive measure ν on S such that $\mathscr{L}^1(\mu) = \mathscr{L}^2(\nu)$, i.e., a function u: $S \to C$ is μ-integrable if and only if $|u|^2$ is ν-integrable; and

$$\left(\int u\, d\mu \,\middle|\, \int v\, d\mu\right) = \int u\bar{v}\, d\nu, \qquad u, v \in \mathscr{L}^1(\mu) \tag{1}$$

(cf. [5; Theorem 5.9], [10; Theorem 24]). The (uniquely determined) bounded positive measure ν satisfying (1) is called the *control measure* of μ.

For a bounded orthogonally scattered vector measure μ on S with values in H, by $\overline{sp}\{\mu\}$ we denote the closed linear subspace in H spanned by the set $\{\mu(A) \mid A \in \mathscr{S}\}$. Recall that

$$\int u\, d\mu \in \overline{sp}\{\mu\}, \qquad u \in \mathscr{L}^1(\mu).$$

By $P_{\overline{sp}\{\mu\}}$ we denote the orthogonal projection of H onto $\overline{sp}\{\mu\}$.

Let μ_1 and μ_2 be two orthogonally scattered bounded vector measures on S with values in H. Recall that μ_1 and μ_2 are *biorthogonal*, if

$$(\mu_1(A) \mid \mu_2(B)) = 0 \qquad \text{for all} \quad A, B \in \mathscr{S},\ A \cap B = \varnothing;$$

furthermore μ_1 is *fully subordinate* to μ_2 if, in addition, $\overline{sp}\{\mu_1\} \subset \overline{sp}\{\mu_2\}$.

Let μ_1 and μ_2 be two bounded orthogonally scattered vector measures on S with values in H. If μ_1 and μ_2 are biorthogonal, then there exists a uniquely determined bounded complex valued measure ν_{12} on S, the *covariance measure* of μ_1 and μ_2, such that

$$\left(\int u\, d\mu_1 \,\middle|\, \int v\, d\mu_2\right) = \int u\bar{v}\, d\nu_{12}, \qquad u \in \mathscr{L}^1(\mu_1), \quad v \in \mathscr{L}^1(\mu_2) \tag{2}$$

(cf. [12; Lemma 1], [10; Theorem 17], [9; Theorem 2.4.11 and its proof]).

The following lemma is a direct consequence of results obtained by Masani [5; Theorem 5.10] and Ressel [12; Theorem 1 and Remark 1, p. 450].

LEMMA 1. *Let μ_k, $k = 0, 1,..., q$, be a family of bounded orthogonally scattered pairwise biorthogonal vector measures on S with values in H. For $j = 1,..., q$ put*

$$\mu_{j,0}(A) = P_{\overline{sp}\{\mu_0\}}(\mu_j(A)), \qquad A \in \mathscr{S}.$$

Then $\mu_{j,0}$ is a bounded orthogonally scattered vector measure on S with values in $\overline{sp}\{\mu_0\}$ such that

(i) *$\mu_{j,0}, \mu_k$, $j = 1,..., q$; $k = 0, 1,..., q$, is a family of $2q + 1$ bounded orthogonally scattered pairwise biorthogonal vector measures;*

(ii) *$\mu_{j,0}$ is fully subordinate to μ_0, $j = 1,..., q$.*

If ν_{00} is the control measure of μ_0 and if ν_{j0} is the covariance measure of μ_j and μ_0, then ν_{j0} is absolutely continuous with respect to ν_{00} and

$$\mu_{j,0}(A) = \int \chi_A \frac{d\nu_{j0}}{d\nu_{00}}\, d\mu_0, \qquad A \in \mathscr{S}, \tag{3}$$

$j = 1,..., q$. Here χ_A is the characteristic function of a set $A \in \mathscr{S}$.

Remark. Suppose the bounded orthogonally scattered vector measures μ and μ' are biorthogonal. Then for any $a, b, c, d \in C$,

$$\mu_1 = a\mu + b\mu' \qquad \text{and} \qquad \mu_2 = c\mu + d\mu'$$

are bounded orthogonally scattered biorthogonal vector measures.

Let μ_j, $j = 1,..., q$, be a family of bounded orthogonally scattered pairwise biorthogonal vector measures on S with values in H. The $q \times q$-matrix

$$\mathbf{C}(\boldsymbol{\mu}) = (\nu_{jk})_{j,k=1}^{q}$$

where ν_{jj} and ν_{jk} are the bounded measures satisfying (1) and (2), respectively, $j, k = 1,..., q$, is called the *covariance matrix* of $\boldsymbol{\mu} = (\mu_1,..., \mu_q)$.

The following theorem is now obvious.

THEOREM 2 (Matveev [7]). *Let μ_j, $j = 0, 1,..., q$, be a family of bounded orthogonally scattered pairwise biorthogonal vector measures on S with values in H and let $\mathbf{C}(\boldsymbol{\mu}) = (\nu_{jk})_{j,k=0}^{q}$ be the covariance matrix of $\boldsymbol{\mu} = (\mu_0,..., \mu_q)$. Put*

$$\mu_{j,0}(A) = P_{\overline{sp}\{\mu_0\}}(\mu_j(A)) = \int \chi_A \frac{d\nu_{j0}}{d\nu_{00}}\, d\mu_0, \qquad A \in \mathscr{S}, \quad j = 1,..., q,$$

$$\tilde{\mu}_0(A) = \mu_0(A), \qquad \tilde{\mu}_0'(A) = 0, \qquad A \in \mathscr{S},$$

$$\tilde{\mu}_j(A) = \mu_{j,0}(A), \qquad \tilde{\mu}_j'(A) = \mu_j(A) - \mu_{j,0}(A), \qquad A \in \mathscr{S}; \quad j = 1,..., q.$$

Then $\tilde{\mu}_j$, $j = 0, 1,\ldots, q$, and $\tilde{\mu}_j'$, $j = 0, 1,\ldots, q$, are families of $q + 1$ bounded orthogonally scattered pairwise biorthogonal vector measures on S with values in H such that

$$\bigvee_{j=0}^{q} \overline{sp}\{\tilde{\mu}_j\} = \overline{sp}\{\mu_0\}, \qquad \bigvee_{j=0}^{q} \overline{sp}\{\mu_j\} = \left(\bigvee_{j=0}^{q} \overline{sp}\{\tilde{\mu}_j\}\right) \oplus \left(\bigvee_{j=0}^{q} \overline{sp}\{\tilde{\mu}_j'\}\right).$$

The covariance matrices of $\tilde{\boldsymbol{\mu}} = (\tilde{\mu}_0,\ldots, \tilde{\mu}_q)$ and $\tilde{\boldsymbol{\mu}}' = (\tilde{\mu}_0',\ldots, \tilde{\mu}_q')$ are $\mathbf{C}(\tilde{\boldsymbol{\mu}}) = (\tilde{\nu}_{jk})_{j,k=0}^{q}$ and $\mathbf{C}(\tilde{\boldsymbol{\mu}}') = (\tilde{\nu}_{jk}')_{j,k=0}^{q}$, respectively, where

$$\tilde{\nu}_{00}(A) = \nu_{00}(A), \qquad \tilde{\nu}_{00}'(A) = 0, \qquad A \in \mathscr{S};$$

$$\tilde{\nu}_{jj}(A) = \int \chi_A \left| \frac{d\nu_{j0}}{d\nu_{00}} \right|^2 d\nu_{00},$$

$$\tilde{\nu}_{jj}'(A) = \nu_{jj}(A) + \tilde{\nu}_{jj}(A) - 2 \operatorname{Re} \nu_{j0}(A), \qquad A \in \mathscr{S}, \quad j = 1,\ldots, q;$$

$$\tilde{\nu}_{jk}(A) = \int \chi_A \frac{d\nu_{j0}}{d\nu_{00}} \overline{\frac{d\nu_{k0}}{d\nu_{00}}}\, d\nu_{00},$$

$$\tilde{\nu}_{jk}'(A) = \nu_{jk}(A) + \tilde{\nu}_{jk}(A) - \int \chi_A \overline{\frac{d\nu_{k0}}{d\nu_{00}}}\, d\nu_{j0} - \int \chi_A \frac{d\nu_{j0}}{d\nu_{00}}\, d\nu_{0k},$$

$$A \in \mathscr{S}; \quad j, k = 0, 1,\ldots, q;$$

$$\tilde{\nu}_{jk}(A) = \overline{\tilde{\nu}_{kj}(A)}, \qquad \tilde{\nu}_{jk}'(A) = \overline{\tilde{\nu}_{kj}'(A)}, \qquad A \in \mathscr{S}; \quad j, k = 0, 1,\ldots, q.$$

EXAMPLE 3. Suppose μ_j, $j = 1,\ldots, q$, is a family of bounded orthogonally scattered pairwise biorthogonal vector measures on S with values in H and let $\mathbf{C}(\boldsymbol{\mu}) = (\nu_{jk})_{j,k=1}^{q}$ be the covariance matrix of $\boldsymbol{\mu} = (\mu_1,\ldots, \mu_q)$. Suppose β is a (possibly unbounded) positive (Radon) measure on S. For all $j = 1,\ldots, q$, there then exists a uniquely determined decomposition $\mu_j = \mu_{j,s} + \mu_{j,c}$, the *Lebesgue decomposition* for μ_j with respect to β, where $\mu_{j,s}$ is a β-singular and $\mu_{j,c}$ is a β-continuous bounded orthogonally scattered vector measure on S with values in $\overline{sp}\{\mu_j\}$. If $\nu_{jj} = \nu_{jj,s} + \nu_{jj,c}$ is the Lebesgue decomposition for ν_{jj} with respect to β, then $\nu_{jj,s}$ and $\nu_{jj,c}$ are the control measures of $\mu_{j,s}$ and $\mu_{j,c}$, respectively. Furthermore,

$$\overline{sp}\{\mu_j\} = \overline{sp}\{\mu_{j,s}\} \oplus \overline{sp}\{\mu_{j,c}\}$$

(cf. [11]). It is clear that

$$\overline{sp}\{\mu_{j,s}\} \perp \overline{sp}\{\mu_{k,c}\}, \qquad j, k = 1,\ldots, q,$$

and, a fortiori, $\mu_j, \mu_{j,s}, \mu_{j,c}, j = 1,\ldots, q$, is a family of $3q$ bounded orthogonally scattered pairwise biorthogonal vector measures. Furthermore,

$$\bigvee_{j=1}^{q} \overline{sp}\{\mu_j\} = \left(\bigvee_{j=1}^{q} \overline{sp}\{\mu_{j,s}\}\right) \oplus \left(\bigvee_{j=1}^{q} \overline{sp}\{\mu_{j,c}\}\right). \tag{4}$$

The covariance matrices of $\boldsymbol{\mu}_s = (\mu_{1,s},\ldots, \mu_{q,s})$ and $\boldsymbol{\mu}_c = (\mu_{1,c},\ldots, \mu_{q,c})$ are

$$\mathbf{C}(\boldsymbol{\mu}_s) = (\nu_{jk,s})_{j,k=1}^{q} \qquad \text{and} \qquad \mathbf{C}(\boldsymbol{\mu}_c) = (\nu_{jk,c})_{j,k=1}^{q},$$

respectively, where $\nu_{jk} = \nu_{jk,s} + \nu_{jk,c}$ is the Lebesgue decomposition for ν_{jk} with respect to β; $j, k = 1,\ldots, q$. The decomposition

$$\mathbf{C}(\boldsymbol{\mu}) = \mathbf{C}(\boldsymbol{\mu}_s) + \mathbf{C}(\boldsymbol{\mu}_c)$$

is the Lebesgue decomposition for $\mathbf{C}(\boldsymbol{\mu})$ with respect to β obtained by Cramér [1; Sect. 4, Theorem 2] (cf. [3; Sect. 1.1], [14; p. 366]).

2. Stationary Processes

In this section we recall the Wold decomposition for a q-variate (weakly stationary) stochastic process.

Let G be a locally compact Abelian group and let H be a complex Hilbert space. Recall that a *q-variate stochastic process* is a function $\mathbf{x}_g = (x_g^1,\ldots, x_g^q)$, $g \in G$, where $x_g^j \in H$, $j = 1,\ldots, q$; $g \in G$. A stochastic process $\mathbf{x}_g = (x_g^1,\ldots, x_g^q)$, $g \in G$, is *q.m. continuous*, if all the mappings $g \to x_g^j$, $g \in G$; $j = 1,\ldots, q$, are continuous. A stochastic process $\mathbf{x}_g$, $g \in G$, is (*weakly*) *stationary*, if the $q \times q$ Gramian matrix

$$(\mathbf{x}_g, \mathbf{x}_h) = ((x_g^j \mid x_h^k))_{j,k=1}^{q} = K(h^{-1}g),$$

depends only on $h^{-1}g$; $h, g \in G$. A stationary stochastic process $\mathbf{x}_g$, $g \in G$, is q.m. continuous, if and only if its *covariance function* $K(g)$, $g \in G$, is continuous on G.

Let $\mathbf{x}_t$, $t \in T$, be a stochastic process defined on a set T and let $\mathscr{J}$ be a family of nonempty subsets in T. By $\overline{sp}\{\mathbf{x}\}$ we denote the closed linear subspace in H spanned by the set $\{x_t^j \mid j = 1,\ldots, q; t \in T\}$. For $J \in \mathscr{J}$, by $\overline{sp}\{\mathbf{x}; J\}$ we denote the closed linear subspace in H spanned by the set $\{x_t^j \mid j = 1,\ldots, q; t \in J\}$. Furthermore we put

$$\overline{sp}\{\mathbf{x}; \mathscr{J}\} = \bigcap_{J \in \mathscr{J}} \overline{sp}\{\mathbf{x}; J\}.$$

The stochastic process $\mathbf{x}_t$, $t \in T$, is called *$\mathscr{J}$-singular*, if $\overline{sp}\{\mathbf{x}; \mathscr{J}\} = \overline{sp}\{\mathbf{x}\}$; it is called *$\mathscr{J}$-regular*, if $\overline{sp}\{\mathbf{x}; \mathscr{J}\} = \{0\}$.

Remark. Tjøstheim [19], [20] has defined the class of regular (or purely non-deterministic) stochastic processes in the case $T = R^n$ by applying so-called innovations of the process. The innovation approach and its connection between the $\mathcal{J}$-regularity has been studied e.g. by Schmidt [16], [17] and by Schmidt and Weron [18] (see also the references given there).

Let $\mathbf{x}_t$, $t \in T$, be a stochastic process and let $\mathcal{J}$ be a family of nonempty subsets in T such that for any $t \in T$ there exists a $J \in \mathcal{J}$ for which $t \in J$. Then

$$\mathbf{y}_t = (P_{\overline{sp}\{\mathbf{x};\mathcal{J}\}}x_t^1, \dots, P_{\overline{sp}\{\mathbf{x};\mathcal{J}\}}x_t^q), \qquad \mathbf{w}_t = \mathbf{x}_t - \mathbf{y}_t, \qquad t \in T,$$

where $P_{\overline{sp}\{\mathbf{x};\mathcal{J}\}}$ is the orthogonal projection of H onto $\overline{sp}\{\mathbf{x}; \mathcal{J}\}$, is the uniquely determined decomposition, the *Wold decomposition for* $\mathbf{x}_t$, $t \in T$, *with respect to* $\mathcal{J}$, satisfying the conditions:

(W.a) $\mathbf{y}_t$, $t \in T$, and $\mathbf{w}_t$, $t \in T$, are q-variate stochastic processes;

(W.b) $\overline{sp}\{\mathbf{y}; J\} \subset \overline{sp}\{\mathbf{x}; J\}$, $\overline{sp}\{\mathbf{w}; J\} \subset \overline{sp}\{\mathbf{x}; J\}$, $J \in \mathcal{J}$; $\overline{sp}\{\mathbf{y}\} \subset \overline{sp}\{\mathbf{x}\}$, $\overline{sp}\{\mathbf{w}\} \subset \overline{sp}\{\mathbf{x}\}$;

(W.c) $\overline{sp}\{\mathbf{y}\} \perp \overline{sp}\{\mathbf{w}\}$;

(W.d) $\mathbf{y}_t$, $t \in T$, is $\mathcal{J}$-singular; $\mathbf{w}_t$, $t \in T$, is $\mathcal{J}$-regular.

If $\mathbf{x}_g$, $g \in G$, is stationary (and q.m. continuous), if $\mathcal{J}$ is closed under translations and if, in addition,

(W.e) the $q \times q$ Gramian matrices $(\mathbf{x}_g, \mathbf{y}_h)$ and $(\mathbf{x}_g, \mathbf{w}_h)$ depend only on $h^{-1}g$; $g, h \in G$;

then even $\mathbf{y}_g$, $g \in G$, and $\mathbf{w}_g$, $g \in G$, are stationary (and q.m. continuous); cf. Salehi and Scheidt [15; Theorem 2.13] and Weron [21; Theorem 2].

LEMMA 4. *Let* $\mathbf{x}_t$, $t \in T$, *be a stochastic process defined on a set* T *and let* $\mathcal{J}$ *be a family of nonempty subsets in* T *such that for any* $t \in T$ *there exists a* $J \in \mathcal{J}$ *for which* $t \in J$. *Suppose* $M \subset \overline{sp}\{\mathbf{x}; \mathcal{J}\}$ *is a closed linear subspace in* H. *Put*

$$\mathbf{z}_t = (P_M x_t^1, \dots, P_M x_t^q), \qquad \mathbf{v}_t = \mathbf{x}_t - \mathbf{z}_t, \quad t \in T,$$

where P_M *is the orthogonal projection of* H *onto* M. *Let* $\mathbf{v}_t = \mathbf{r}_t + \mathbf{u}_t$, $t \in T$, *be the Wold decomposition for* $\mathbf{v}_t$, $t \in T$, *w.r.t.* $\mathcal{J}$. *Then*:

(i) $\mathbf{z}_t$, $t \in T$, *is* $\mathcal{J}$-*singular*;

(ii) $\overline{sp}\{\mathbf{z}\} \subset \overline{sp}\{\mathbf{x}\}$, $\overline{sp}\{\mathbf{v}\} \subset \overline{sp}\{\mathbf{x}\}$;

(iii) $\overline{sp}\{\mathbf{z}\} \perp \overline{sp}\{\mathbf{v}\}$;

(iv) $\overline{sp}\{\mathbf{x}; \mathcal{J}\} = M \oplus \overline{sp}\{\mathbf{v}; \mathcal{J}\}$;

(v) $\mathbf{x}_t = \mathbf{y}_t + \mathbf{w}_t$, $t \in T$, *where*

$$\mathbf{y}_t = \mathbf{z}_t + \mathbf{r}_t, \qquad \mathbf{w}_t = \mathbf{u}_t, \quad t \in T,$$

is the Wold decomposition for $\mathbf{x}_t$, $t \in T$, *w.r.t.* $\mathscr{J}$.

Proof. (i) Since $M \subset \overline{sp}\{\mathbf{x}; \mathscr{J}\}$,

$$\mathbf{z}_t = (P_M x_t^1,..., P_M x_t^q) = (P_M y_t^1,..., P_M y_t^q), \qquad t \in T.$$

The $\mathscr{J}$-singularity of $\mathbf{z}_t$, $t \in T$, follows then from the $\mathscr{J}$-singularity of $\mathbf{y}_t$, $t \in T$, since any q-variate stochastic process that is a bounded linear transformation of a $\mathscr{J}$-singular q-variate stochastic process is $\mathscr{J}$-singular.

(ii)–(iii) The assertions (ii)–(iii) follow immediately from the definition of $\mathbf{z}_t$, $t \in T$.

(iv) Since $M \subset \overline{sp}\{\mathbf{x}; \mathscr{J}\}$, it is clear that $\overline{sp}\{\mathbf{v}; J\} \subset \overline{sp}\{\mathbf{x}; J\}$ for all $J \in \mathscr{J}$. Thus, $\overline{sp}\{\mathbf{v}; \mathscr{J}\} \subset \overline{sp}\{\mathbf{x}; \mathscr{J}\}$ and, a fortiori, $M \oplus \overline{sp}\{\mathbf{v}; \mathscr{J}\} \subset \overline{sp}\{\mathbf{x}; \mathscr{J}\}$.

On the other hand, suppose $z \in \overline{sp}\{\mathbf{x}; \mathscr{J}\}$. Put $z = z_1 + z_2$, where $z_1 \in M$ and $z_2 \in \overline{sp}\{\mathbf{x}; \mathscr{J}\}$, $z_2 \perp M$. In order to show that $z_2 \in \overline{sp}\{\mathbf{v}; \mathscr{J}\}$, note that for any $\epsilon > 0$ and for any element $z' \in \overline{sp}\{\mathbf{x}; J\}$, $J \in \mathscr{J}$,

$$z' = \sum_{k=1}^{n} a_k x_{t(k)}^{j(k)},$$

$t(k) \in J$, $1 \leqslant j(k) \leqslant q$; $k = 1,..., n$, satisfying $\| x - z' \| < \epsilon$ one has

$$\|(I - P_M)(z - z')\| < \epsilon$$

and

$$(I - P_M)(z - z') = z_2 - \sum_{k=1}^{n} a_k v_{t(k)}^{j(k)}.$$

Thus, the fact that $z \in \overline{sp}\{\mathbf{x}; \mathscr{J}\}$ implies that $z_2 \in \overline{sp}\{\mathbf{v}; \mathscr{J}\}$, proving the assertion (iv).

(v) The assertion (v) follows immediately from (iv).

The lemma is proved.

Remark. Let $\mathbf{x}_g$, $g \in G$, be a stationary (and q.m. continuous) stochastic process and let U_g: $\overline{sp}\{\mathbf{x}\} \to \overline{sp}\{\mathbf{x}\}$, $g \in G$, be the shift operator group of $\mathbf{x}_g$, $g \in G$, i.e., the group of unitary operators on $\overline{sp}\{\mathbf{x}\}$ for which $U_g x_h^j = x_{gh}^j$; $g, h \in G$, $j = 1,..., q$. Suppose $\mathscr{J}$ is a family of nonempty subsets in G, which is closed under translations. If $M \subset \overline{sp}\{\mathbf{x}; \mathscr{J}\}$ is a closed linear subspace in H, which is invariant under U_g, $g \in G$, then even the stochastic processes

$$\mathbf{z}_g = (P_M x_g^1, \dots, P_M x_g^q), \qquad \mathbf{v}_g = \mathbf{x}_g - \mathbf{z}_g, \quad g \in G,$$

are stationary (and q.m. continuous).

EXAMPLE 5. (a) Let $\mathbf{x}_t = (x_t^1, \dots, x_t^q)$, $t \in T$, be a stochastic process defined on a set T and let $\mathscr{J}$ be a family of nonempty subsets in T such that for any $t \in T$ there exists a $J \in \mathscr{J}$ for which $t \in J$. Then

$$\overline{sp}\{x^j; \mathscr{J}\} \subset \overline{sp}\{\mathbf{x}; \mathscr{J}\}, \qquad j = 1, \dots, q,$$

and, a fortiori, the stochastic processes

$$\mathbf{z}_t = (P_M x_t^1, \dots, P_M x_t^q), \qquad \mathbf{v}_t = \mathbf{x}_t - \mathbf{z}_t, \quad t \in T, \tag{5}$$

have the properties (i)–(v) stated in Lemma 4 for any M of the form

$$M = \bigvee_{j \in K} \overline{sp}\{x^j; \mathscr{J}\}, \qquad K \subset \{1, \dots, q\}. \tag{6}$$

Note that the inclusion

$$\bigvee_{j=1}^{q} \overline{sp}\{x^j; \mathscr{J}\} \subset \overline{sp}\{\mathbf{x}; \mathscr{J}\}$$

may be strict (cf. Example 9).

(b) Suppose $\mathbf{x}_g = (x_g^1, \dots, x_g^q)$, $g \in G$, is a (q.m. continuous) stationary stochastic process and suppose $\mathscr{J}$ is a family of nonempty subsets in G, which is closed under translations. Then all the closed linear subspaces $\overline{sp}\{x^j; \mathscr{J}\}$, $j = 1, \dots, q$, in H are invariant under the shift operator group U_g, $g \in G$, of $\mathbf{x}_g$, $g \in G$. Thus, the stochastic processes $\mathbf{z}_g$, $g \in G$, and $\mathbf{v}_g$, $g \in G$, defined in (5) are stationary (and q.m. continuous) having the properties (i)–(v) stated in Lemma 4 for any M of the form (6).

(c) Suppose $\mathbf{x}_g = (x_g^1, \dots, x_g^q)$, $g \in G$, is a q.m. continuous stationary stochastic process. Then there exists a family of bounded orthogonally scattered pairwise biorthogonal vector measures $\boldsymbol{\mu} = (\mu_1, \dots, \mu_q)$, the spectral measure of $\mathbf{x}_g$, $g \in G$, defined on the dual group Γ of G such that $\overline{sp}\{\mu_j\} = \overline{sp}\{x^j\}$,

$$x_g^j = \int \overline{\gamma(g)}\, d\mu_j(\gamma), \qquad g \in G, \quad j = 1, \dots, q.$$

Let $\mathscr{J}$ be a family of nonempty subsets in G, which is closed under translations. Suppose one of the component processes of $\mathbf{x}_g$, $g \in G$, say $x_g{}^1$, $g \in G$, is $\mathscr{J}$-singular. Put $M = \overline{sp}\{x^1\}$ $(=\overline{sp}\{x^1; \mathscr{J}\})$. Then the stochastic processes $\mathbf{z}_g$, $g \in G$, and $\mathbf{v}_g$, $g \in G$, defined in (6) are q.m. continuous and stationary stochastic processes having the properties (i)–(v) stated in Lemma 4. Furthermore,

$$z_g{}^j = \int \overline{\gamma(g)}\, d\left(\frac{d\nu_{j1}}{d\nu_{11}} \cdot \mu_1\right)(\gamma), \qquad v_g{}^j = \int \overline{\gamma(g)}\, d\left(\mu_j - \frac{d\nu_{j1}}{d\nu_{11}} \cdot \mu_1\right)(\gamma),$$
$$g \in G, \quad j = 1, \dots, q, \qquad (7)$$

where $\mathbf{C}(\boldsymbol{\mu}) = (\nu_{jk})_{j,k=0}^{q}$ is the covariance matrix of $\boldsymbol{\mu}$.

Recall that the covariance function $K(g)$, $g \in G$, of $\mathbf{x}_g$, $g \in G$, has a spectral representation in the form

$$(x_g{}^j \mid x_h{}^k) = \int \overline{\gamma(h^{-1}g)}\, d\nu_{jk}(\gamma); \qquad g, h \in G; \quad j, k = 1, \dots, q.$$

The spectral representations for the covariance functions of $\mathbf{z}_g$, $g \in G$, and $\mathbf{v}_g$ $g \in G$, respectively, can be obtained by applying Theorem 2.

Stationary stochastic processes obtained by the orthogonal projection formula (7) have been considered by Matveev [7].

3. On the Construction of the Wold Decomposition

In this section we consider the construction of the Wold decomposition connected with the classical linear prediction problem for a q-variate stationary stochastic process. For convenience we consider here only the case $G = Z$ even if our results can be transformed into the case $G = R$. In our case $\mathscr{J} = \mathscr{J}_{-\infty} = \{(-\infty, k] \subset Z \mid k \in Z\}$.

We show first that in forming the Wold decomposition w.r.t. $\mathscr{J}_{-\infty}$, we can restrict ourselves into the case of stationary stochastic processes with spectral measures that are absolutely continuous with respect to m, the Lebesgue measure of $(0, 2\pi]$.

Theorem 6. *Let $\mathbf{x}_k$, $k \in Z$, be a stationary stochastic process and let $\boldsymbol{\mu} = (\mu_1, \dots, \mu_q)$ be its spectral measure. Let $\boldsymbol{\mu}_s = (\mu_{1,s}, \dots, \mu_{q,s})$ and $\boldsymbol{\mu}_c = (\mu_{1,c}, \dots, \mu_{q,c})$ be the m-singular and m-continuous parts of $\boldsymbol{\mu}$, respectively. Then*

$$M = \bigvee_{j=1}^{q} \overline{sp}\{\mu_{j,s}\} \subset \overline{sp}\{\mathbf{x}; \mathscr{J}_{-\infty}\};$$

and the stationary stochastic processes $\mathbf{z}_k$, $k \in Z$; $\mathbf{v}_k$, $k \in Z$,

$$z_k^j = \int e^{ik\lambda}\, d\mu_{j,s}(\lambda), \qquad v_k^j = \int e^{ik\lambda}\, d\mu_{j,c}(\lambda), \qquad k \in Z, \quad j = 1,..., q,$$

have the properties (i)–(v) *stated in Lemma* 4.

Proof. The theorem follows from Lemma 4, since it is well-known that

$$\overline{sp}\{z^j\} = \overline{sp}\{\mu_{j,s}\} \subset \overline{sp}\{\mathbf{x}; \mathscr{I}_{-\infty}\}, \qquad j = 1,..., q,$$

and

$$\overline{sp}\{\mathbf{x}\} = \bigvee_{j=1}^{q} \overline{sp}\{x^j\} = \bigvee_{j=1}^{q} \overline{sp}\{\mu_j\} = \left(\bigvee_{j=1}^{q} \overline{sp}\{\mu_{j,s}\}\right) \oplus \left(\bigvee_{j=1}^{q} \overline{sp}\{\mu_{j,c}\}\right)$$

(cf. Example 3).

Suppose the spectral measure $\boldsymbol{\mu} = (\mu_1, ..., \mu_q)$ of a stationary stochastic process $\mathbf{x}_k$, $k \in Z$, is m-continuous, or equivalently, the covariance matrix $\mathbf{C}(\boldsymbol{\mu}) = (\nu_{jk})_{j,k=1}^{q}$ of $\boldsymbol{\mu}$ consist of m-continuous measures. The $q \times q$ matrix of functions $\mathbf{f} = (f_{jk})_{j,k=1}^{q}$,

$$f_{jk} = \frac{d\nu_{jk}}{dm}, \qquad j, k = 1,..., q,$$

is called the spectral density of $\mathbf{x}_k$, $k \in Z$.

Recall that the *rank* $r(\mathbf{x})$ of a stationary stochastic process $\mathbf{x}_k$, $k \in Z$, is defined to be the rank of the $q \times q$ Gramian matrix $(\mathbf{x}_0 - \hat{\mathbf{x}}_0, \mathbf{x}_0 - \hat{\mathbf{x}}_0)$, where

$$\hat{\mathbf{x}}_0 = (P_{\overline{sp}\{\mathbf{x};(-\infty,-1]\}} x_0^1, ..., P_{\overline{sp}\{\mathbf{x};(-\infty,-1]\}} x_0^q).$$

Here $P_{\overline{sp}\{\mathbf{x};(-\infty,-1]\}}$ is the orthogonal projection of H onto $\overline{sp}\{\mathbf{x}; (-\infty, -1]\}$.

Remark 7. (a) Suppose $\mathbf{x}_k = \mathbf{u}_k + \mathbf{v}_k$, $k \in Z$, is a decomposition for a q-variate stationary stochastic process $\mathbf{x}_k$, $k \in Z$, such that

(R.1) $\mathbf{u}_k$, $k \in Z$, and $\mathbf{v}_k$, $k \in Z$, are q-variate stationary stochastic processes;

(R.2) $\overline{sp}\{\mathbf{u}\} \perp \overline{sp}\{\mathbf{v}\}$, $\overline{sp}\{\mathbf{u}\} \subset \overline{sp}\{\mathbf{x}\}$, $\overline{sp}\{\mathbf{v}\} \subset \overline{sp}\{\mathbf{x}\}$.

Then $r(\mathbf{x}) = r(\mathbf{u}) + r(\mathbf{v})$, if and only if

$$S(\mathbf{x}) = S(\mathbf{u}) + S(\mathbf{v}), \qquad R(\mathbf{x}) = R(\mathbf{u}) + R(\mathbf{v}),$$

where $\mathbf{x} = S(\mathbf{x}) + R(\mathbf{x})$, $\mathbf{u} = S(\mathbf{u}) + R(\mathbf{u})$, $\mathbf{v} = S(\mathbf{v}) + R(\mathbf{v})$ are the corresponding Wold decompositions w.r.t. $\mathscr{I}_{-\infty}$. Especially,

$$S(\mathbf{x}) = \mathbf{u}, \qquad R(\mathbf{x}) = \mathbf{v},$$

if and only if $r(\mathbf{x}) = r(\mathbf{v})$ and $\mathbf{v}_k$, $k \in Z$, is $\mathscr{I}_{-\infty}$-regular; cf. Robertson [13; Corollary 2.7, Theorem 3.1], Jang Ze-Pei [2; Part I, Theorem 9]. (For the case $G = R$ cf. [13; Sect. 6]).

(b) Recall that Wiener and Masani [22; Theorem 7.12] and Matveev [6], [8] (cf. Masani [4; pp. 371–372]) have presented necessary and sufficient conditions for a stationary stochastic process $\mathbf{x}_k$, $k \in Z$, to be $\mathscr{I}_{-\infty}$-regular with $r(\mathbf{x}) = q$ and $r(\mathbf{x}) = \rho$, $0 < \rho < q$, respectively. (For the case $G = R$ cf. [8]).

(c) Suppose a stationary stochastic process $\mathbf{x}_k$, $k \in Z$, has an m-continuous spectral measure and suppose its spectral density $\mathbf{f}$ satisfies the condition

$$\operatorname{rank} \mathbf{f} \leqslant 1 \qquad \text{a.e. } (m).$$

Recall that in this case $\mathbf{x}_k$, $k \in Z$, is either $\mathscr{I}_{-\infty}$-singular or $\mathscr{I}_{-\infty}$-regular (cf. Robertson [13; Lemma 4.3, Corollary 5.3]).

(d) For (necessary and) sufficient conditions for a stationary stochastic process $\mathbf{x}_k$, $k \in Z$, with an m-continuous spectral measure to be $\mathscr{I}_{-\infty}$-singular see [2; Part I, Theorem 15 and Part II], [7].

We are now ready to present an algorithm, which gives a partial solution to the problem how to construct the Wold decomposition for a stationary stochastic process $\mathbf{x}_k$, $k \in Z$, provided that its spectral measure is known. In general, even $r(\mathbf{x})$ must be known. Unfortunately the algorithm does not give a complete solution to this problem (cf. Example 9(b)). Notice the algorithmic notation.

ALGORITHM. Let $\mathbf{x}_k$, $k \in Z$, be a stationary stochastic process and let $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_q)$ be its spectral measure.

For any stationary stochastic process $\mathbf{z}_k$, $k \in Z$, by $S(\mathbf{z})$ and $R(\mathbf{z})$ we denote the $\mathscr{I}_{-\infty}$-singular and $\mathscr{I}_{-\infty}$-regular parts of $\mathbf{z}_k$, $k \in Z$, respectively.

Step 1. Let $\boldsymbol{\mu}_s$ and $\boldsymbol{\mu}_c$ be the m-singular and m-continuous parts of $\boldsymbol{\mu}$, respectively. Put

$$u_k^j = \int e^{ik\lambda}\, d\mu_{j,s}(\lambda), \qquad v_k^j = \int e^{ik\lambda}\, d\mu_{j,c}(\lambda), \qquad k \in Z, \quad j = 1, \ldots, q.$$

Step 2. If $R(\mathbf{v}) = 0$, then

$$S(\mathbf{x}) = \mathbf{x}, \qquad R(\mathbf{x}) = 0$$

(cf. Example 5(c), Theorem 6, Remark 7(d)).

If $R(\mathbf{v}) \neq 0$ (or if it cannot be decided whether $R(\mathbf{v}) \neq 0$ or not), go to Step 3.

Step 3. If $S(\mathbf{v}) = 0$, then

$$S(\mathbf{x}) = \mathbf{u}, \qquad R(\mathbf{x}) = \mathbf{v}$$

(cf. Example 5(c), Theorem 6, Remark 7(b)).
If $S(\mathbf{v}) \neq 0$, go to Step 4.

Step 4. If there does not exist any (non-zero) $\mathscr{J}_{-\infty}$-singular component v_k^j, $k \in Z$, of $\mathbf{v}_k$, $k \in Z$, go to Step 5.

If there exists a (non-zero) $\mathscr{J}_{-\infty}$-singular component, say v_k^1, $k \in Z$, of $\mathbf{v}_k$, $k \in Z$, put

$$w_k^j = \int e^{ik\lambda} \frac{d\tilde{\nu}_{j1}}{d\tilde{\nu}_{11}}(\lambda)\, d\tilde{\mu}_1(\lambda), \qquad k \in Z, \quad j = 1,\dots, q, \tag{8}$$

where $\tilde{\boldsymbol{\mu}} = (\tilde{\mu}_1, \dots, \tilde{\mu}_q)$ is the spectral measure of $\mathbf{v}_k$, $k \in Z$, and $\mathbf{C}(\tilde{\boldsymbol{\mu}}) = (\tilde{\nu}_{jk})_{j,k=1}^q$ is the covariance matrix of $\tilde{\boldsymbol{\mu}}$. Put

$$\mathbf{u}_k = \mathbf{u}_k + \mathbf{w}_k, \qquad \mathbf{v}_k = \mathbf{v}_k - \mathbf{w}_k, \quad k \in Z,$$

and go to Step 2.

Step 5. Choose, if possible, a (non-zero) component, say v_k^1, $k \in Z$, of $\mathbf{v}_k$, $k \in Z$, such that for the stationary stochastic processes $\mathbf{w}_k$, $k \in Z$, defined in (8), and $\mathbf{w}_k' = \mathbf{v}_k - \mathbf{w}_k$, $k \in Z$, one has

$$r(\mathbf{v}) = r(\mathbf{w}) + r(\mathbf{w}'). \tag{9}$$

Then $r(\mathbf{w}) = 0$ or $r(\mathbf{w}) = 1$ (cf. Remark 7(d)). If $r(\mathbf{w}) = 0$, put

$$\mathbf{u}_k = \mathbf{u}_k + \mathbf{w}_k, \qquad \mathbf{v}_k = \mathbf{v}_k - \mathbf{w}_k, \quad k \in Z,$$

and go to Step 2 (cf. Remark 7(a)). If $r(\mathbf{w}) = 1$, then

$$S(\mathbf{x}) = \mathbf{u} + S(\mathbf{w}'), \qquad R(\mathbf{x}) = \mathbf{w} + R(\mathbf{w}') \tag{10}$$

(cf. Remark 7(a)).
If no components v_k^j, $k \in Z$, of $\mathbf{v}_k$, $k \in Z$, satisfying (9) can be found, then

$$S(\mathbf{x}) = \mathbf{u} + S(\mathbf{v}), \qquad R(\mathbf{x}) = R(\mathbf{v}). \tag{11}$$

Remark 8. (a) In order to construct the Wold decomposition $\mathbf{w}' = S(\mathbf{w}') + R(\mathbf{w}')$ in (10) one can try to apply the algorithm. However, the Wold decomposition $\mathbf{v} = S(\mathbf{v}) + R(\mathbf{v})$ in (11) cannot be formed by applying the algorithm (cf. Example 9(b)).

(b) In some special cases the use of the algorithm can be impoved by noting that for the rank $r(\mathbf{x})$ and the spectral density $\mathbf{f}$ of a stationary stochastic process $\mathbf{x}_k$, $k \in Z$, with an m-continuous spectral measure, one has

$$r(\mathbf{x}) \leqslant \operatorname{essinf} \operatorname{rank}(\mathbf{f}) \quad (m); \qquad \text{and} \qquad r(\mathbf{x}) \geqslant \max(r(\mathbf{u}), r(\mathbf{v}))$$

for any stationary stochastic processes $\mathbf{u}_k$, $k \in Z$, $\mathbf{v}_k$, $k \in Z$, such that

$$\mathbf{x}_k = \mathbf{u}_k + \mathbf{v}_k, \qquad k \in Z; \qquad \overline{sp}\{\mathbf{u}\} \perp \overline{sp}\{\mathbf{v}\}$$

[13; Theorem 2.1 and Sect. 5] (cf. Example 9(a)).

EXAMPLE 9. (a) Consider two stationary stochastic processes $y_{j,k}$, $k \in Z$, $j = 1, 2$, with m-continuous spectral measures such that $\overline{sp}\{y_1\} \perp \overline{sp}\{y_2\}$ and

$$(y_{j,h} \mid y_{j,k}) = \int e^{i(h-k)\lambda} f_j(\lambda)\, d\lambda, \qquad h, k \in Z, \quad j = 1, 2,$$

where

$$f_1 = 1, \qquad f_2(\lambda) = \begin{cases} 0, & \lambda \in (0, \pi] \\ 1, & \lambda \in (\pi, 2\pi], \end{cases}$$

i.e., $y_{1,k}$, $k \in Z$, is $\mathscr{I}_{-\infty}$-regular and $y_{2,k}$, $k \in Z$, is $\mathscr{I}_{-\infty}$-singular.

Define a 2-variate stationary stochastic process $\mathbf{x}_k = (x_k^1, x_k^2)$, $k \in Z$, by

$$x_k^1 = y_{1,k} + y_{2,k}, \qquad x_k^2 = y_{1,k}, \quad k \in Z.$$

Then

$$S(\mathbf{x}) = (y_2, 0), \qquad R(\mathbf{x}) = (y_1, y_1). \tag{12}$$

When applying the algorithm, in order to obtain (12), notice that the spectral density of $\mathbf{x}_k$, $k \in Z$, is

$$\mathbf{f} = \begin{pmatrix} f_1 + f_2 & f_1 \\ f_1 & f_1 \end{pmatrix}.$$

Clearly, $\mathbf{f}$ does not satisfy the conditions given by Wiener and Masani [23] (cf. [8]), in order $\mathbf{x}_k$, $k \in Z$, to be $\mathscr{I}_{-\infty}$-regular with $r(\mathbf{x}) = 2$ or $r(\mathbf{x}) = 1$. Thus, $r(\mathbf{x}) \leqslant 1$. Since x_k^j, $k \in Z$, are $\mathscr{I}_{-\infty}$-regular we must apply Step 5 with $\mathbf{u} = 0$, $\mathbf{v} = \mathbf{x}$. The orthogonal projection onto $\overline{sp}\{v^2\}$ gives

$$\mathbf{w} = (y_1, y_1), \qquad \mathbf{w}' = (y_2, 0),$$

i.e., $r(\mathbf{w}) = 1$, $r(\mathbf{w}') = 0$. Since $r(\mathbf{v}) \leqslant 1$ and $r(\mathbf{v}) \geqslant \max(r(\mathbf{w}), r(\mathbf{w}'))$ (cf. Remark 8(b)), we get

$$r(\mathbf{v}) = r(\mathbf{w}) + r(\mathbf{w}'),$$

i.e.,

$$S(\mathbf{x}) = \mathbf{w}' = (y_2, 0), \qquad R(\mathbf{x}) = \mathbf{w} = (y_1, y_1).$$

(b) Let $y_{j,k}$, $k \in Z$; $j = 1, 2$, be defined as in (a). Define a 2-variate stationary stochastic process $\mathbf{x}_k$, $k \in Z$, by

$$x_k^1 = y_{1,k} + y_{2,k}, \qquad x_k^2 = y_{1,k} + 2y_{2,k}, \quad k \in Z.$$

Then

$$S(\mathbf{x}) = (y_2, 2y_2), \qquad R(\mathbf{x}) = (y_1, y_1).$$

In order to apply the algorithm notice that the spectral density $\mathbf{f}$ of $\mathbf{x}_k$, $k \in Z$, is

$$\mathbf{f} = \begin{pmatrix} f_1 + f_2 & f_1 + 2f_2 \\ f_1 + 2f_2 & f_1 + 4f_2 \end{pmatrix}.$$

As in (a) we must apply Step 5 with $\mathbf{u} = 0$, $\mathbf{v} = \mathbf{x}$. But in this case the orthogonal projection onto $\overline{sp}\{v^1\}$ and $\overline{sp}\{v^2\}$, respectively, produces in both cases stationary stochastic processes $\mathbf{w}_k$, $k \in Z$, and $\mathbf{w}'_k$, $k \in Z$, that are $\mathscr{J}_{-\infty}$-singular and, a fortiori, different from $S(\mathbf{x})$ and $R(\mathbf{x})$. Thus, in this case the algorithm does not give a complete solution. The algorithm indicates the failure with the inequality

$$1 = r(\mathbf{x}) > r(\mathbf{w}) + r(\mathbf{w}') = 0.$$

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